VARIETIES OF INTERIOR ALGEBRAS

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Key results in this paper have subsequently been published with new exposition in "Naturman, Colin Ashley. Interior algebras and topology. Diss. University of Cape Town, 1991, Chap 1 Basic Theory of Interior Algebras"

This work includes additional expository material ancillary to the above.

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Abstract

An interior algebra is a Boolean Algebra enriched with an interior operator. The structure of the lattice of interior algebra varieties is investigated. Results concerning equational bases for interior algebra varieties are discussed. The concept of a Heyting variety is introduced and used to show that there are $2^{\aleph_0}$ interior algebra varieties which are not generated by their finite members.

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INTRODUCTION

An interior algebra is a Boolean algebra enriched wiith an interior oooperator. (It is essentially the same as a closure algebra.) [7] contains the basic results concerning interior algebras and we assume that the reader is familiar with these. In section 1 of this paper we investigate the structure of the lattice of interior algebra varieties. Some results concerning equational bases of interior algebra varieties are discussed in section 2, and in section 3 we introduce the concept of a Heyting variety and show that there are $2^{\aleph_0}$ Heyting varieties of interior algebras which are not generated by their finite members. Throughout this paper we will make implicit use of results concerning congruence distributivity; the standard reference for these is [4].

NOTATION AND TERMINOLOGY

We use the same notational conventions and terminology as in [7]. In additions: If B is an S.I. interior algebra then the largest non–1 element of B will be referred to as the monolith of B and will be denoted m_B. If $\mathcal{V}$ is a variety $\Lambda_{\mathcal{V}}$ will denote the lattice of subvarieties of $\mathcal{V}$. $\mathcal{V}_{DI}$, $\mathcal{V}_{SI}$ and $\mathcal{V}_{FSI}$ will denote the classes of directly indecomposable (D.I.), subdirectly irreducible (S.I.) and finitely subdirectly irreducible (F.S.I.) members of $\mathcal{V}$ respectively. If λ is a cardinal then $\mathcal{F}(\mathcal{V}, \lambda)$ denotes the free algebra in $\mathcal{V}$ on λ generators. □

1. The Lattice of Interior Algebra Varieties.

Let $\mathcal{I}$ denote the variety of all interior algebras and let $\Lambda = \Lambda_{\mathcal{I}}$.

As usual Λ is a dually algebraic lattice in which the dual compact elements are the finitely based varieties.

$\mathcal{I}$ is the top element of Λ and $\mathcal{T}$ is the bottom element, where $\mathcal{Y}$ is the variety of all trivial interior algebras (in which $0 = 1$).

Interior algebras are congruence distributive so we have:

Corollary 1.1

Λ is a distributive lattice. □

Lemma 1.2 (see [7])

$\mathcal{I}$ is generated by its finite S.I. members. □

Proposition 1.3

Every proper subvariety of $\mathcal{I}$ has a cover in Λ.

Proof:

Let $\mathcal{V}$ be a proper subvariety of $\mathcal{I}$. By Lemma 1.2 there is a finite S.I. interior algebra B such that $B \notin \mathcal{V}$. Then the interval $\mathcal{V} + \mathrm{Var}\{B\}/\mathcal{V}$ in Λ is finite and so the result follows. □

Lemma 1.4

Let $\mathcal{V}$ be a congruence distributive variety such that $\mathcal{V} = SH\ \mathcal{V}_{SI}$. Then $\mathcal{V}$ is join irreducible in $\Lambda_{\mathcal{V}}$

Proof:

Let $\mathcal{W}$, $\mathcal{Z}$ be proper subvarieties of $\mathcal{V}$. Then there are algebras $M \in \mathcal{V} \setminus \mathcal{W}$ and $N \in \mathcal{V} \setminus \mathcal{Z}$. Then $M \times N \notin \mathcal{W}$ and $M \times N \notin \mathcal{Z}$. Since $\mathcal{V} = \mathrm{SH}\ \mathcal{V}_{SI}$ there is a $K \in \mathcal{V}_{SI}$ such that $M \times N \in \mathrm{SH}\{K\}$. Then $K \notin \mathcal{W}$ and $K \notin \mathcal{Z}$. By congruence distributivity $(\mathcal{W} + \mathcal{Z})_{SI} = \mathcal{W}_{SI} \cup \mathcal{Z}_{SI}$ and so $K \notin (\mathcal{W} + \mathcal{Z})_{SI}$. Hence $\mathcal{W} + \mathcal{Z} \neq \mathcal{V}$. Thus $\mathcal{V}$ is join irreducible in $\Lambda_{\mathcal{V}}$ □

Lemma 1.5

Let B be an interior algebra.

(i) B is embeddable in an epimorphic image of a complete atomic S.I. interior algebra.

(ii) If B is finite then B is an epimorphic image of a finite S.I. interior algebra.

Proof:

Let $X = T(B) \cup \{T(B)\}$ and let $\tau = Y(B) \cup \{X\}$. Consider the topological space $\mathbf{X} = \langle X, \tau \rangle$. The inclusion map $i : T(B) \to \mathbf{X}$ is an open embedding and so $Ai : A(\mathbf{X}) \to AT(B)$ is an epimorphism. Notice that $\mathbf{X}$ is supercompact and so $A(\mathbf{X})$ is S.I. $A(\mathbf{X})$ is also complete and atomic. Now B is embeddable in $AT(B)$ and so (i) follows. If B is finite then B is in fact isomorphic to $AT(B)$ and $A(\mathbf{X})$ is finite. Hence (ii) follows. □

Theorem 1.6

$\mathcal{I}$ is join irreducible in Λ, but not completely join irreducible.

Proof:

By lemma 1.5 $\mathcal{I} = \mathrm{SH}\ \mathcal{I}_{SI}$. Hence by lemma 1.4 $\mathcal{I}$ is join irreducible in Λ. Now suppose $\mathcal{I}$ is completely join irreducible in Λ. Then there is a largest proper subvariety $\mathcal{V}$ of $\mathcal{I}$. Since $\mathcal{V} \neq \mathcal{I}$ by lemma 1.2 there is a finite S.I. interior algebra B such that $B \notin \mathcal{V}$. Then $\mathcal{V}$ is a proper subvariety of $\mathcal{V} + \mathrm{Var}\{B\}$ and so $\mathcal{I} = \mathcal{V} + \mathrm{Var}\{B\}$. Since $\mathcal{I}$ is join irreducible $\mathcal{I} = \mathrm{Var}\{B\}$. By congruence distributivity and the fact that B is finite we have $\mathcal{I}_{SI}$ is finite which is false (see [7]). Thus $\mathcal{I}$ is not completely join irreducible. □

Thus in fact $\mathcal{I}$ has no dual cover, since by join irreducibility any dual cover would have to be a largest proper subvariety making $\mathcal{I}$ completely join irreducible.

Recall that there are only two Boolean algebra varieties: the variety of all Boolean algebras and the variety of all trivial Boolean algebras (in which $0 = 1$). Thus if σ is an interior algebra identity which holds in some non–trivial interior algebra and does not contain ^{I}, then σ holds in all interior algebras. Hence interior algebra varieties are distinguished by identities involving ^{I}.

Definition 1.7

An interior algebra is said to be Boolean iff it satisfies the identity $x^{I} = x$. □

Corollary 1.8

Let X be a topological space. Then $A(X)$ is Boolean iff X is discrete.

Let $\mathcal{B}$ denote the variety of all Boolean interior algebras. We can identify $\mathcal{B}$ with the variety of all Boolean algebras. Up to isomorphism there is only one two element interior algebra viz. **2**, the two element Boolean algebra with both elements open. Then $\mathcal{B}_{SI} = \{\mathbf{2}\}$. If B is a non–trivial interior algebra then $0 \neq 1$ in B and so $\{0, 1\} \subseteq B$ forms a subalgebra isomorphic to **2**. We thus have:

Theorem 1.9

B is the smallest subvariety above $\mathscr{T}$. □

We also see that if σ is an interior algebra identity which fails in 2 then σ fails in all non–trivial interior algebras.

If $\mathscr{K}$ is a class of topological spaces let $\mathscr{V}(\mathscr{K})$ denote the variety $\mathrm{Var}\,\{A(X) : X \in \mathscr{K}\}$. If $\mathscr{K} = \{X_0, \dots, X_n\}$ let $\mathscr{V}(X_0, \dots, X_n)$ denote $\mathscr{V}(\mathscr{K})$.

For any cardinal λ let I_λ denote the indiscrete space on λ . I_0 is the unique empty space and $I_1 \cong T(2)$. Recall that for $\lambda > 0$ $A(I_\lambda)$ is simple (see [7]).

Lemma 1.10

Let $0 < n < \omega$. Then $S\{A(I_n)\} = \{A(I_k) : 0 , k \leq n\}$.

Proof:

Let $0 , k \leq n$. Then there is a surjective map $f\colon I_n \to I_k$. Then f is also a continuous open map. Hence $Af\colon A(I_k) \to A(I_n)$ is an embedding. Conversely suppose B is embeddable in $A(I_n)$. Then B is simple with $|B| = 2^k$ where $0 < k \leq n$. Then $T(B)$ is indiscrete with k elements. Hence $T(B) \cong I_k$ and so $B \cong AT(B) \cong A(I_k)$. □

Lemma 1.11

Let $0 < n < \omega$. Then $\mathscr{V}(I_n)_{SI} = \{A\,(I_k) : 0 < k \leq n\}$.

Proof:

$S\{A(I_n)\} = \{A(I_k) : 0 < k \leq n\} \subseteq \mathscr{V}(I_n)_{SI}$ by lemma 1.10. By congruence distributivity $\mathscr{V}(I_n) \subseteq HS\{A(I_n)\}$. Since for $0 < k \leq n$ $A(I_k)$ is simple the result follows. □

Theorem 1.12

Λ contains a chain of varieties: $\mathscr{V}(I_0) \subset \mathscr{V}(I_1) \subset \mathscr{V}(I_2) \subset \mathscr{V}(I_3)$ ...
where for all $n < \omega$ $\mathscr{V}(I_{n+1})$ is a (completely) join irreducible cover of $\mathscr{V}(I_n)$.

Proof:

Note that $\mathscr{V}(I_0) = \mathscr{T}$. By lemma 1.11 we see that for all $n < \omega$: $\mathscr{V}(I_{n+1}) = \mathscr{V}(I_n)_{SI} \cup \{A(I_{n+1})\}$ and so the result follows. □

Lemma 1.13

Let B be an interior algebra such that L(B) is finite. Then B is locally finite.

Proof:

Let $L(B) = \{a_0, \dots, a_n\}$. Then for every interior algebra polynomial $p(x_0, \dots, x_n)$ there and $b_0, \dots, b_k \in B$ there is a Boolean algebra polynomial $q(x_0, \dots, x_n, y_0, \dots, y_n)$ such that $p[b_0. \dots, b_k] = q[b_0, \dots, b_k, a_0, \dots, a_n]$ in B . Since all Boolean algebras are locally finite the result follows. □

Obviously the converse of the above lemma does not hold: consider an infinite Boolean interior algebra.

Let $\mathscr{C} = \sum\limits_{n<\omega} \mathscr{V}(I_n)$

Proposition 1.14
Let B be an infinite simple interior algebra. Then $\mathscr{C} = \mathrm{Var}\{B\}$.

Proof:

Let $0 < n < w$. Then since B is infinite there is an embedding $f\colon \mathrm{Ba}\,A(I_n) \to \mathrm{Ba}\,B$. Now $LA(I_n) = \{\phi , n\}$ and $L(B) = \{0 , 1\}$. Thus f is in fact an interior algebra embedding $f\colon A(I_n) \to B$. Hence $\mathscr{C} \subseteq \mathrm{Var}\{B\}$.

Let σ be an interior algebra identity with m variables. Suppose $B \nvDash \sigma$. Then there is a subset $K \subseteq B$ with $|K| \leq m$ and $C \models \sigma$ where C is the subalgebra of B generated by K . Then C is finite by lemma 1.13 and simple. Hence $C \cong A(I_n)$ for some n with $0 < n < \omega$. Thus $A(I_n) \models \sigma$. Hence $\mathrm{Id}\,\mathscr{C} \subseteq \mathrm{Id}\,\{B\}$ and so $\mathrm{Var}\{B\} \subseteq \mathscr{C}$. □

We have interesting characterizations of the members of $\mathscr{C}$. Let S denote the Sierpinski space.

Theorem 1.15
Let B be an interior algebra. The following are equivalent:

(i) $B \in \mathscr{C}$

(ii) $B \models (x^{IC} = x^I)$

(iii) A(S) is not embeddable in B

(iv) $L(B) = L'(B)$.

Proof:

(i) ⟹ (ii) :

By proposition 1.14 $\mathscr{C} = \mathscr{V}(I_{\aleph_0})$. Clearly $A(I_{\aleph_0}) \models (x^{IC} = x^I)$ and so if $B \in \mathscr{C}$ then $B \models (x^{IC} = X^I)$.

(ii) ⟹ (iii):

Let $B = (x^{IC} = x^I)$. Suppose there is an embedding $f\colon A(S) \to B$. Then $f(\{0\})^I = f(\{0\})$ and so $f(\{0\})^C = f(\{0\})$. But $f(\{0\})^C = f(\{0\}^C = f(\{0,1\}) = 1 \neq f(\{0\})$, a contradiction.

(iii) ⟹ (iv)

Suppose $L(B) \neq L'(B)$. Then L(B) is not closed under complementation and so there is a $b \in L(B)$ with $b'^I < b'$. Put $c = b + b'^I$. Now $b > 0$ and so $c > 0$. Also $b'^I \geq b'$ and so $c < 1$. Now $b, b'^I \in L(B)$ and so $c \in L(B)$, $c'^I = (b'b'^{I'})^I = b'^{I'}$. $b'^{I'I'} \leq b'^{I'} = 0$. Thus $\{0, c, c', I\}$ forms a subalgebra of B isomorphic to A(S) , and so the result follows by contraposition.

(iv) ⟹ (i):

Let B be such that $L(B) = L'(B)$. Now B is isomorphic to a subdirect product of its S.I. epimorphic images. Let $f\colon B \to C$ be an epimorphism where C is S.I. Then $Lf\colon L(B) \to L(C)$ is an epimorphism. Let $a \in L(C)$. Then there is a $b \in L(B) = L'(B)$ such that $f(b) = Lf(b) = a$. Hence $a \in L'(C)$. Thus $L(C) \subseteq L'(C)$ and so $L(C) = \mathrm{CoC}$. Since C is S.I. 1 is completely join irreducible in L(C) and hence in CoC. Thus CoC must be the two element chain $\{0, 1\}$. Since $L(C) = \mathrm{CoC}$, C is simple. Hence $C \in \mathscr{C}$ and so $B \in \mathrm{Ps}\,\mathscr{C} = \mathscr{C}$ as required. □

The above theorem shows :

(i) $\mathscr{C}$ is a finitely based subvariety of $\mathscr{I}$. In fact $\mathscr{C} = \mathscr{I} \cap \mathrm{Mod}\ \{x^{IC} = x^I\}$.

(ii) The identity $x^{IC} = x^I$ is characterized by the exclusion of A(S) .

(iii) $B \in \mathscr{C}$ iff the open elements of B are exactly the closed elements and hence exactly the clopen elements.

Corollary 1.16

$\mathscr{C}_{DI}$ is the class of all simple interior algebras.

Proof:

If $B \in \mathscr{C}_{DI}$ then $L(B) = CoB = \{0, 1\}$ and so B is simple. Clearly if B is simple $B \in \mathscr{C}_{DI}$. □

Notice that in the varieties $\mathscr{V}(I_n)$, $n < \omega$ and $\mathscr{C}$ the concepts of simple, S.I., F.S.I. and D.I. coincide. In fact we have :

Corollary 1.17

(i) $\Lambda_{\mathscr{C}} = \{\mathscr{V}(I_n) : n < \omega\} \cup \{\mathscr{C}\}$.

(ii) $\Lambda_{\mathscr{C}}$ consists precisely of the semi–simple varieties in Λ.

(iii) $\Lambda_{\mathscr{C}}$ consists precisely of the discriminator varieties in Λ.

Proof:

(i) and (ii) are clear from Lemma 1.11, Proposition 1.14 and Corollary 1.16. For (iii) note that $(xy + x'y')^I z + (x'y + xy')^C x$ represents the discriminator function in all simple interior algebras. Thus the varieties in $\Lambda_{\mathscr{C}}$ are all discriminator varieties. Since all discriminator varieties are semi–simple the result follows. □

Corollary 1.18

$(\mathscr{C}, \mathscr{V}(S))$ is a splitting pair in Λ. $A(S)$ is a splitting interior algebra with conjugate identity $x^{IC} = x^I$. □

Corollary 1.19

Let $\mathscr{V} \in \Lambda$ with $\mathscr{B} \subset \mathscr{V}$. Then $\mathscr{V}(I_2) \subseteq \mathscr{V}$ or $\mathscr{V}(S) \subseteq \mathscr{V}$.

Proof:

By Corollary 1.18 $\mathscr{V} \subseteq \mathscr{C}$ or $\mathscr{V}(S) \subseteq \mathscr{V}$. In the former case we have $\mathscr{V}(I_2) \subseteq \mathscr{V}$ by Corollary 1.17(i) and the fact that $\mathscr{V}(I_1) = \mathscr{B} \subset \mathscr{V}$. □

Thus in particular $\mathscr{V}(I_2)$ and $\mathscr{V}(S)$ are the unique covers of $\mathscr{B}$ in Λ.

Notice that 2 and $A(S)$ are both splitting interior algebras [3] and they are both finite. Since $\mathscr{J}$ is congruence distributive and generated by its finite S.I. members it follows by the same argument as for lattices that every splitting algebra in $\mathscr{J}$ is finite. We give a general proof for completeness.

Proposition 1.20

Let $\mathscr{V}$ be a congruence distributive variety generated by its finite (S.I.) members. Then every splitting algebra in $\mathscr{V}$ is finite.

Proof:

Let $M \in \mathscr{V}$ be splitting. Then $Var\{M\}$ is completely join prime in $\Lambda_{\mathscr{V}}$. Now $\mathscr{V}$ is generated by its finite (S.I.) members so $Var\{M\}$ is generated by some finite S.I. algebra N [6]. By congruence distributivity we must have $M \cong N$ and so M is finite. □

Any projective S.I. algebra is splitting [3]. 2 is clearly projective, we also have :

Proposition 1.21

$A(S)$ is projective.

Proof:

Let $f: B \to A(S)$ be an epimorphism. Then $Lf: L(B) \to LA(S)$ is an epimorphism and so there is a $b \in L(B)$ with $Lf(b) = \{0\}$. Put $c = b + b'^{I}$. Then $c \in L(B)$. Also $c'^{I} = (b'.b'^{I\prime})^{I} = b'^{I}.b'^{I\prime} = 0$, and $f(c) = f(b) + f(b)'^{I} = \{0\} + \{0\}'^{I} = \{0\}$. It follows that $\{0, c, c', 1\}$ forms a subalgebra of B isomorphic to $A(S)$ whose inclusion map is a co–retraction of f. □

If B is a non–simple S.I. interior algebra then by Theorem 1.15 $A(S)$ is embeddable in B. In fact we have :

Proposition 1.22

Let B be non–simple and S.I. Then $\{0, m_B, m_B', 1\}$ forms a subalgebra of B isomorphic to $A(S)$.

Proof:

Since B is non–simple $m_B > 0$. Then $m_B'^{I} \leq m_B' < 1$ and so $m_B'^{I} \leq m_B$. Hence $m_B'^{I} = m_B'^{I}\, m_B \leq m_B'\, m_B = 0$ and the result follows. □

We now investigate covers in Λ. The covers of $\mathscr{V}(I_n)$, $2 \leq n < \omega$ are easy to determine, in fact we have :

Proposition 1.23

Let $2 \leq n < \omega$. $\mathscr{V}(I_{n+1})$ is the unique join irreducible cover of $\mathscr{V}(I_n)$ and $\mathscr{V}(I_n, S)$ in the unique join reducible cover of $\mathscr{V}(I_n)$.

Proof:

Let $\mathscr{V} \in \Lambda$ with $\mathscr{V}(I_n) \subset \mathscr{V}$. By Corollary 1.18 $\mathscr{V} \subseteq \mathscr{C}$ on $\mathscr{V}(S) \subseteq \mathscr{V}$. Since $\mathscr{V}(I_n) \subset \mathscr{V}$, in the former case $\mathscr{V}(I_{n+1}) \subseteq \mathscr{V}$ by Corollary 1.17(i) and in the latter case $\mathscr{V}(I_n, S) \subseteq \mathscr{V}$ whence the result follows. □

Definition 1.24

Define the following supercompact topological spaces.

$X_0 = \langle 3, \{\phi, \{0\}, 3\}\rangle$

$X_1 = \langle 3, \{\phi, \{0\}, \{0,1\}, 3\}\rangle$

$X_2 = \langle 3, \{\phi, \{0\}, \{1\}, \{0,1\}, 3\}\rangle$

$X_3 = \langle 3, \{\phi, \{0,1\}, 3\rangle$. □

In the following lemmas B will denote a non–simple S.I. interior algebra and m will denote the monolith m_B of B. Since B is non–simple $m > 0$.

Lemma 1.25

If m is not a co–atom then $A(X_0)$ is embeddable in B.

Proof:

Since m is not a co–atom there is a $b \in B$ with $m < b < 1$. $\{0, m, m', b, b', m'b. m+b', 1\}$ forms an eight element Boolean algebra, we show that it in fact forms a subalgebra of B isomorphic to $A(X_0)$. We have $m = m^{I} \leq (m+b')^{I} \leq m$ i.e. $(m+b')^{I} = m$. By Proposition 1.22 $m'^{I} = 0$. Since $m'b, b' \leq m'$ we also have $(m'b)^{I} = b'^{I} = 0$. □

Lemma 1.26

Suppose m is a co–atom and B/m is D.I. but non–simple. Then $A(X_1)$ is embeddable in B.

Proof:

B/m is non–simple so there is a $b \in L(B)$ with $0 < b < m$. $b'^{I} \leq m$ and $bb'^{I} = 0$

hence $b + b'^{I} \neq m$ or else $b \in \mathrm{Co}\, B/m$ contradicting the fact that B/m is D.I. Put $c = b + b'^{I}$. Then $0 < c < m$ and $c \in L(B)$. Then $\{0, m, m', c, c', mc', m' + c, 1\}$ forms an eight element Boolean algebra, we show that it forms a subalgebra of B isomorphic to $A(X_1)$. We have $c'^{I} = (b'b'^{I\prime})^{I} = b'^{I} b'^{I\prime I} = 0$. Then also $(mc')^{I} = 0$. By Proposition 1.22 $m'^{I} = 0$. Now $m' + c \notin L(B)$ or else $m' \leq m' + c \leq m$ a contradiction. Thus $(m' + c)^{I} < m' + c$. m is a co–atom and so m' is an atom. Hence $m' + c$ covers c and so $(m' + c)^{I} = c$. □

Lemma 1.27

Suppose m is a co–atom and B/m is not D.I. Then $A(X_2)$ is embeddable in B.

Proof:

B/m is not D.I. and so there is a $b \in C_0\, B/m$ with $0 < b < m$. Then $\{0, m, m', b, b', mb', m' + b, 1\}$ forms an eight element Boolean algebra, we show that it forms a subalgebra of B isomorphic to $A(X_2)$. $b, mb' \in L(B)$. $b' \notin L(B)$ or else $1 = b + b' \leq m$ a contradiction. Hence $b'^{I} < b'$. Now m is a co–atom and so m' is an atom. Hence $b' = m' + mb'$ covers mb'. Hence $b' = m' + mb'$ covers mb'. Hence $b'^{I} = mb'$. Now $m' + b \notin L(B)$ or else $m' \leq m' + b \leq m$ a contradiction. Hence $(m' + b)^{I} < m' + b$. But $m' + b$ covers b so $(m' + b)^{I} = b$. By Proposition 1.22 $m'^{I} = 0$. □

Lemma 1.28

Suppose m is a co–atom, $L(B) = \{0, m, 1\}$ and $B \cong A(S)$. Then $A(X_3)$ is embeddable in B.

Proof:

Since $B \cong A(S)$ there is a $b \in B$ with $0 < b < m$. If $a \in B$ is not open then $0 < a < 1$ but $m \leq a$ since m is a co–atom. Hence $a^{I} = 0$. Thus $\{0, m, m', b, b', mb', m' + b, 1\}$ forms a subalgebra of B isomorphic to $A(X_3)$. □

Theorem 1.29

$\mathscr{V}(X_0)$, $\mathscr{V}(X_1)$ and $\mathscr{V}(X_2)$ are precisely the join irreducible covers of $\mathscr{V}(S)$ and $\mathscr{V}(I_2,S)$ is the unique join reducible cover of $\mathscr{V}(S)$.

Proof:

Clearly $\mathscr{V}(I_2,S)$ is the unique join reducible cover of $\mathscr{V}(S)$, and $\mathscr{V}(X_i)$ or $i = 0, 1, 2$ are overs of $\mathscr{V}(S)$. To see that they are the only join irreducible covers consider $\mathscr{V} \in \Lambda$ with $\mathscr{V}(S) \subset \mathscr{V}$ but $\mathscr{V}(I_2) \subseteq \mathscr{V}$. Then there is a $B \in \mathscr{V}_{SI}$ with $B \cong 2$ and $B \cong A(S)$. B is non–simple or else $A(I_2)$ is embeddable in B whence $\mathscr{V}(I_2) \subseteq \mathscr{V}$ a contradiction. If m_B is not a co–atom then by Lemma 1.25 $\mathscr{V}(X_0) \subseteq \mathscr{V}$. Consider the case where m_B is a co–atom. If B/m_B is not D.I. then by Lemma 1.27 $\mathscr{V}(X_2) \subseteq \mathscr{V}$. Consider the case where B/m_B is D.I. Suppose B/m_B is simple. Then $B/m_B \cong 2$ or else $B \cong A(S)$ a contradiction. Hence $A(I_2) \subseteq \mathscr{V}$ a contradiction. Thus B/m_B is non–simple and so by Lemma 1.26 $\mathscr{V}(X_1) \subseteq \mathscr{V}$. □

Theorem 1.30

$\mathscr{V}(I_2,X_i)$ for $i = 0, 1, 2$ and $\mathscr{V}(I_3,S)$ are precisely the join reducible covers of $\mathscr{V}(I_2,S)$ and $\mathscr{V}(X_3)$ is the unique join irreducible cover of $\mathscr{V}(I_2,S)$.

Proof:

That the join reducible covers are as specified is clear. $\mathscr{V}(X_3)$ is also clearly a cover of

$\mathscr{V}(I_2,S)$. To see that it is the unique join irreducible cover consider $\mathscr{V} \in \Lambda$ with $\mathscr{V}(I_2,S \subset \mathscr{V}$ but $\mathscr{V}(X_i) \subseteq \mathscr{V}$ for $i = 0, 1, 2$ and $\mathscr{V}(I_3) \subseteq \mathscr{V}$. Then there is a $B \cong \mathscr{V}_{SI}$ with $B \cong 2$, $B \cong A(BI_2)$ and $B \cong A(S)$. B is non–simple or else $A(I_3)$ is embeddable in B whence $\mathscr{V}(I_3) \subseteq \mathscr{V}$ a contradiction. Then m_B must be a co–atom or else by Lemma 1.25 $\mathscr{V}(X_0) \subseteq \mathscr{V}$ a contradiction. B/m_B is D.I. or else by Lemma 1.27 $\mathscr{V}(X_2) \subseteq \mathscr{V}$ a contradiction. We must then have that B/m_B is simple or else by Lemma 1.26 $\mathscr{V}(X_1) \subseteq \mathscr{V}$ again a contradiction. Hence $L(B/m_B) = \{0,m_B\}$ and so $L(B) = \{0,m_B,1\}$. Then by Lemma 1.28 $\mathscr{V}(X_3) \subseteq \mathscr{V}$. □

Definition 1.31

For $\lambda > 0$ define $S_\lambda = \langle \lambda, \{\phi, \{0\}, \lambda\}\rangle$. □

For $\lambda > 0$ S_λ is a supercompact topological space. $S_1 \cong I_1$, $S_2 \cong S$ and $S_3 \cong X_0$. We can use these spaces to obtain a chain of join irreducible covers in Λ .

Lemma 1.32

Let $0 < n < \omega$. Then $S\{A(S_n)\} = \{A(S_k): 0 < k \leq n\}$.

Proof:

Let $0 < k \leq n$. Then there is a surjective map $f: S_n \to S_k$ with $f(0) = 0$. Then f is also an open continuous map. Hence $Af: A(S_k) \to A(S_n)$ is an embedding. Conversely suppose $g: B \to A(S_n)$ is an embedding. Then $Lg: L(B) \to LA(S_n)$ is an embedding. Hence $L(B) = \{0, 1\}$ or $L(B) = \{0, a, 1\}$ where $g(a) = \{0\}$. Consider $L(B) = \{0, 1\}$. Suppose there is a $b \in B$ with $0 < b < 1$. Then $g(b)^I = g(b^I) = g(0) = \phi$. Hence $\{0\} \subseteq g(b)$ and so $\{0\} \subseteq g(b)' = g(b')$. Now $0 < b' < 1$. Hence $g(b') \neq n$ and so $g(b')^I = \{0\}$. But $g(b')^I = g(b'^I) = g(0) = \phi$, a contradiction. Hence $B \cong 2 \cong A(S_1)$.

Consider $L(B) = \{0, a, 1\}$ where $S(a) = \{0\}$. Then a is an atom in B . Hence $\mathscr{T}(B) = \{\phi, \{[a)\}, T(B)\}$ and so $T(B) \cong S_k$ for some k with $1 < k \leq n$. Hence $B \cong AT(B) \cong A(S_k)$, □

Lemma 1.33

Let $0 < n < \omega$. Then $\mathscr{V}(S_n)_{SI} = \{A(S_k): 0 < k \leq n\}$.

Proof:

$S\{A(S_n)\} = \{A(S_k): 0 < k \leq n\} \subseteq \mathscr{V}(S_n)_{SI}$ by Lemma 1.32. By congruence distributivity $\mathscr{V}(S_n) \subseteq HS\{A(S_n)\}$. Let $0 < k \leq n$. Then the only epimorphic images of $A(S_k)$ are the trivial interior algebra, $2 \cong A(S_1)$ and $A(S_k)$, hence the result follows. □

Theorem 1.34

Λ contains a chain of varieties : $\mathscr{T} \subset \mathscr{V}(S_1) \subset \mathscr{V}(S_2) \subset \mathscr{V}(S_3) \subset \ldots$
where for all $0 < n < \omega$ $\mathscr{V}(S_{n+1})$ is a completely join irreducible cover of $\mathscr{V}(S_n)$.

Proof:

Note that $\mathscr{V}(S_1) = \mathscr{B}$. If $0 < n < \omega$ then by Lemma 1.33 $\mathscr{V}(S_{n+1})_{SI} = \mathscr{V}(S_n)_{SI} \cup \{A(S_{n+1})\}$ and so the result follows. □

Let $\mathscr{S} = \sum_{0<n<w} \mathscr{V}(S_n)$.

Proposition 1.35

Let B be an infinite interior algebra such that $L(B) = \{0, a, 1\}$ where a is an atom in B . Then $\mathscr{S} = \mathrm{Var}\{B\}$.

Proof:

Let $0 < n < \omega$. We show that $A(S_n)$ is embeddable in B. For $n = 1$ this is clear so consider $n > 1$. Then $\{0\}$ is an atom in $A(S_n)$ and $\{0\} \neq n$. Then since B is infinfite there is a Boolean algebra embedding $f\colon BaA(S_n) \to BaB$. Moreover we can choose f such that $f(\{0\}) = a$. Since $LA(S) = \{0, \{0\}, n\}$ and $L(B) = \{0, a, 1\}$. It follows that f is in fact an interior algebra embedding $f\colon A(S_n) \to B$. Thus $\mathscr{S} \subseteq \mathrm{Var}\,\{B\}$.

Now let σ be an interior algebra identity with m variables. Suppose $B \models \sigma$. Then there is a subset $K \subseteq B$ such that $|K| \leq m$ and $C \models \sigma$ where C is the subalgebra of B generated by K. By Lemma 1.13 C is finite. If $C \cong 2 \cong A(S_1)$ then $A(S_1) \models \sigma$. If $C \cong 2$ then there is a $b \in K$ such that $0 < b < 1$. Then since a is an atom $a \leq b$ or $a \leq b'$. Hence $b^I = a$ or $b'^I = a$ and so $a \in C$. Hence $L(C) = \{0, a, 1\}$. Since a is an atom in B, a is an atom in C. Arguing as in Lemma 1.32 $BC \cong A(S_n) \models \sigma$. Hence $\mathrm{Id}\,\mathscr{S} \subseteq \mathrm{Id}\,\{B\}$ and so $\mathrm{Var}\,\{B\} \subseteq \mathscr{S}$. □

Proposition 1.36

For $B \in \mathscr{S}$ the following are equivalent :

(i) B is S.I.

(ii) B is F.S.I.

(iii) $L(B) = \{0, a, 1\}$ where a is an atom B.

Proof:

Clearly (i) and (ii), and (iii) implies (i). It remains to show that (ii) implies (iii). Let φ be the sentence $(\exists x)(x^I = x \wedge \neg(x = 0) \wedge (\forall y)(y \leq x \Rightarrow (y = 0 \vee y = x)))$. Then $B \models \varphi$ iff (iii) holds. Let $\mathscr{K}$ be the class of all interior algebras which satisfy φ. Assume (ii). By Proposition 1.35 $\mathscr{S} = \mathrm{Var}\,\{A(S_{\aleph_0})\}$ say, and so by congruence distributivity and (ii) $B \in HS\,Pu\,\{A(S_{\aleph_0})\}$. Now $A(S_{\aleph_0}) \in \mathscr{K}$ and so $B \in HS\mathscr{K}$. By a similar argument to the one in Lemma 1.32 $S\mathscr{K} \subseteq \mathscr{K}$. Thus $B \in H\mathscr{K}$. If $C \in \mathscr{K}$ then the only epimorphic images of C are the trivial interior algebra, 2 and C. Now $2, C \in \mathscr{K}$ so $H\mathscr{K} = \mathscr{K} \cup \mathscr{T}$. Since B is non–trivial, $B \in \mathscr{K}$ and so (iii) holds. □

From Lemma 1.33, Proposition 1.35 and Proposition 1.36 we have :

Corollary 1.34

$\Lambda\,\mathscr{S} = \{A(S_n)\colon 0 < n < \omega\} \cup \{\mathscr{S}\}$. □

Notive that the varieties we have found are either finitely generated or have a proper class of S.I. members up to isomorphism. We in fact have :

Proposition 1.35

Let $\mathscr{V}$ be a congruence distributive variety such that $\mathscr{V}_{SI}$ or $\mathscr{V}_{FSI}$ is an elementary class. Then the following are equivalent :

(i) $\mathscr{V}$ is finitely generated.

(ii) $\mathscr{V}$ is residually small i.e. $\mathscr{V}_{SI}$ is a set.

(iii) All members of $\mathscr{V}_{SI}$ are finite.

(iv) $\mathscr{V}_{SI}$ is finite.

(v) $\mathscr{V}_{SI}$ is a finite set of finite algebras.

(vi) Any one of (ii) – (v) with $\mathscr{V}_{SI}$ replaced by $\mathscr{V}_{FSI}$.

Proof:

We will assume that $\mathscr{V}_{SI}$ is elementary, the proof for $\mathscr{V}_{FSI}$ elementary is obtained by an analogous argument. By congruence distributivity (i) implies (ii). Suppose $\mathscr{V}_{SI}$ contains an infinite member. Then by the Lowenheim–Skolem–Tarski Theorem $\mathcal{V}_{SI}$ contains arbitrarily large members. Hence (ii) implies (iii). Assume (iii). Suppose $\mathcal{V}_{SI}$ is infinite. Then $\mathcal{V}_{SI}$ contains arbirarily large finite members and hence it contains an infinite member, a contradiction. Hence (iii) implies (iv). Clearly (iv) implies (i). Thus (i)–(iv) are equivalent. From (iii) and (iv) we see that in fact (i)–(v) are equivalent. By congruence distributivity (i) implies (ii)–(iv) with $\mathcal{V}_{SI}$ replaced by $\mathcal{V}_{FSI}$. Thus (i)–(vi) are equivalent. □

In particular the above theorem applies to any $\mathcal{V} \in \Lambda$. If $\mathcal{V} \in \Lambda$ is finitely generated then by (vi) and (iii) above we see that all members of $\mathcal{V}_{FSI}$ are finite and so $\mathcal{V}_{FSI} = \mathcal{V}_{SI}$.

2. EQUATIONAL BASES FOR INTERIOR ALGEBRA VARIETIES.

Theorem 2.1

Let $\mathcal{V} \subseteq \mathcal{W}$ in Λ. The following are equivalent :

(i) $\mathcal{V}$ is finitely based relative to $\mathcal{W}$

(ii) $\mathcal{V}$ is one–based relative to $\mathcal{W}$

(iii) $\mathcal{V}_{FSI}$ is a strictly elementary class relative to $\mathcal{W}$ defined by a universal sentence.

(iv) $\mathcal{V}_{FSI}$ is a strictly elementary class relative to $\mathcal{W}$.

Proof:

Interior algebras are congruence distributive and congruence permutable and so (i) implies (ii) [8]. There is a universal sentence φ defining F.S.I. interior algebras. Assume (ii). Then there is an identity σ defining $\mathcal{V}$. Regarding σ as a universal sentence, $\sigma \wedge \varphi$ is a universal sentence defining $\mathcal{V}_{FSI}$. Thus (ii) implies (iii). Obviously (iii) implies (iv). By congruence distributivity (iv) implies (i) [5]. □

Corollary 2.2

Let $\mathcal{V}, \mathcal{W} \subseteq \mathcal{Z}$ in Λ. If $\mathcal{V}$, $\mathcal{W}$ are finitely based relative to $\mathcal{Z}$ then so is $\mathcal{V} + \mathcal{W}$.

Proof:

By Theorem 2.1 $\mathcal{V}_{FSI}$ and $\mathcal{W}_{FSI}$ are strictly elementary classes relative to $\mathcal{Z}$. Hence so is $\mathcal{V}_{FSI} \cup \mathcal{W}_{FSI}$. But by congruence distributivity $(\mathcal{V} + \mathcal{W})_{FSI} = \mathcal{V}_{FSI} \cup \mathcal{W}_{FSI}$ and so the result follows from Theorem 2.1. □

By congruence distributivity we immediately have from Baker's Theorem [1] :

Proposition 2.3

Let $\mathcal{V} \subseteq \mathcal{W}$ in Λ. If $\mathcal{V}$ is finitely generated then $\mathcal{V}$ is finitely based relative to $\mathcal{W}$. □

Definition 2.4

If p and q are interior algebra polynomials let $p * q$ denote the polynomial $(pq + p'q')^I$. If B is an interior algebra and $a,b \in B$, $a * b$ will thus denote the element $(ab + a'b')^I$ of B. □

The following summarizes the useful properties of $*$.

Proposition 2.5

Let B be an interior algebra and $\psi \in \mathrm{Con}(B)$. Let $a, b \in B$.

(i) $a * b \in L(B)$.

(ii) $a \,\psi\, b$ iff $(a * b) \,\psi\, 1$.

(iii) $F\psi = \{c * d : c, d \in B \text{ and } c \,\psi\, d\}$.

(iv) $\mathrm{Con}(a,b) = \theta_{[a * b]} = \mathrm{Con}(a * b, 1)$. □

Proposition 2.6

Let B be an interior algebra and let $a, b, c, d \in B$. Let $\sigma(x, y, z, w)$ be the identity $z(a * y) = w(x * y)$. Then

$$(c, d) \in \mathrm{Con}(a, b) \text{ iff } B \models \sigma[a, b, c, d].$$ □

The above proposition shows that interior algebras have equationally definable principal congruences in fact definable by a single identity.

Definition 2.7

If λ is a cardinal, let $W(\lambda)$ denote the word algebra (of interior algebra polynomials) on λ. If $\mathcal{V} \in \Lambda$ define a congruence relation $\equiv_{\mathcal{V}}$ on $W(\lambda)$ by $p \equiv_{\mathcal{V}} q$ iff $(p + q) \in \mathrm{Id}\,\mathcal{V}$. If $p \in W(\lambda)$ let $[p]_{\mathcal{V}}$ denote the equivalence class of p under $\equiv_{\mathcal{V}}$, if $\mathcal{V}$ is understood we will simply write $[p]$. Recall that we can consider $\mathcal{F}(\mathcal{V},\lambda) = W(\lambda) / \equiv_{\mathcal{V}}$.

If $\mathcal{W} \subseteq \mathcal{V}$ in Λ let $F(\mathcal{V}, \mathcal{W}, \lambda)$ denote the set $\{[p * q]_{\mathcal{V}} : p, q \in W(\lambda) \text{ and } \equiv_{\mathcal{W}} q\}$.

□

Using the above definitions and Proposition 2.6 we can re–express the connection between fully invariant congruences and subvarieties in terms of full filters and equational bases.

Theorem 2.8

Let λ be a cardinal and $\mathcal{V} \in \Lambda$.

(i) Let $\mathcal{W} \in \Lambda$ with $\mathcal{W} \subseteq \mathcal{V}$ and let $h: \mathcal{F}(\mathcal{V},\lambda) \to \mathcal{F}(\mathcal{W},\lambda)$ be the canonical epimorphism. Then $F(\mathcal{V}, \mathcal{W},\lambda) = F_{\ker h}$.
Consequently $F(\mathcal{V}, \mathcal{W},\lambda)$ is a full filter in $L(\mathcal{F}(\mathcal{V},\lambda))$.

(ii) Let F be a full filter in $L(\mathcal{F}(\mathcal{V},\lambda))$. Then $F = F(\mathcal{V}, \mathcal{W},\lambda)$ where $\mathcal{W}$ is the variety with equational base $\{p = 1 : [p]_{\mathcal{V}} \in F\}$ relative to $\mathcal{V}$. □

Corollary 2.9

Let $\mathcal{W} \subseteq \mathcal{V}$ in Λ and let λ be a cardinal.

Then $\mathcal{F}(\mathcal{V},\lambda) / \theta_{F(\mathcal{V}, \mathcal{W},\lambda)} \cong \mathcal{F}(\mathcal{V},\lambda)$. □

Definition 2.10

Let B be an interior algebra and $S \subseteq L(B)$. Let F be the smallest full filter in $L(B)$ such that $S \subseteq F$. F is said to be **generated** by S, and S is said to be a **base** for F.

A full filter generated by a singleton is said to be **one–based**. □

Corollary 2.11

Let λ be a cardinal and $\mathcal{W} \subseteq \mathcal{V}$ in Λ. Suppose Γ is an equational base for $\mathcal{W}$ relative to $\mathcal{V}$ with variables from λ. Then $\{[p * q]_{\mathcal{V}} : (p = q) \in \Gamma\}$ is a base for $F(\mathcal{V}, \mathcal{W},\lambda)$.

Conversely if S is a base for $F(\mathcal{V}, \mathcal{W},\lambda)$ then $\{p = 1 : [p]_{\mathcal{V}} \in S\}$ is an equational base for $\mathcal{W}$ relative to $\mathcal{V}$. □

Note that $F(\mathcal{V}, \mathcal{W}, \lambda)$ is one–based iff it has a finite base. Thus we obtain an alternative proof of (i) $\Leftrightarrow$ (ii) in Theorem 2.1.

Corollary 2.12

Let $\mathcal{W} \subseteq \mathcal{V}$ in Λ. Then $\mathcal{W}$ is finitely based relative to $\mathcal{V}$ iff $F(\mathcal{V}, \mathcal{W}, \aleph_0)$ is one–based. □

If S is a base for a full filter F in $L(\mathcal{F}(\mathcal{V},\lambda))$, then F is not simply the filter generated by S as a subbase. In particular a one–based filter need not be principal. We can overcome any difficulties caused by this.

Definition 2.13

Let λ be a cardinal and $\mathcal{V} \in \Lambda$. Define a relation $\vdash$ on $W(\lambda)$ as follows :
$p \vdash q$ iff for all $\mathcal{W} \in \Lambda$ with $\mathcal{W} \subseteq \mathcal{V}$, $p \equiv_{\mathcal{W}} 1$ implies $q \equiv_{\mathcal{W}} 1$. Equivalently
$p \vdash q$ iff $\mathrm{Id}\, \mathcal{V} \cup \{p = 1\} \vdash q = 1$.

If it is not true that $p \vdash q$ we write $p \not\vdash q$. □

Proposition 2.14

(i) $[p] \leq [q]$ implies $p \vdash q$

(ii) $p \vdash q$ and $q \vdash r$ implies $p \vdash r$

(iii) $p \vdash qr$ iff $p \vdash q$ and $p \vdash r$

(iv) If $p \vdash q$ and q' is obtained from q by uniform replacement of variables in q by polynomials then $p \vdash q'$. □

Proposition 2.15

Let $F \subseteq L(\mathcal{F}(\mathcal{V},\lambda))$ be a full filter. If $[p] \in F$ and $[q] \in L(\mathcal{F}(\mathcal{V},\lambda))$ with $p \vdash q$ then $[q] \in F$. □

Proposition 2.16

Let $S \subseteq L(\mathcal{F}(\mathcal{V},\lambda))$. Then the full filter generated by S is $\{[p] \in L(\mathcal{F}(\mathcal{V},\lambda)) :$ there are $q_1, \ldots, q_n \in W(\lambda)$ such that $[q_1], \ldots, [q_n] \in S$ and $\prod_{i=1}^{n} q_i \vdash p\}$. □

Corollary 2.17

Let F be a full filter in $L(\mathcal{F}(\mathcal{V},\lambda))$. Then F is one–based iff there is a $q \in W(\lambda)$ such that $F = \{[p] \in L(\mathcal{F}(\mathcal{V},\lambda)) : q \vdash p\}$. □

We can use the relation $\vdash$ to characterize splitting algebras in interior algebra varieties.

Definition 2.18

Let $\mathcal{V} \in \Lambda$ and let $B \in \mathcal{V}_{SI}$. Let λ be a cardinal and $p \in W(\lambda)$. p is said to be a **monolithic polynomial** for B iff there is an epimorphism $f\colon W(\lambda) \to B$ such that $f(p) = m_B$ and if $q \in W(\lambda)$ with $f(q) = m_B$ then $q \vdash p$. □

Theorem 2.19

Let $\mathcal{V} \in \Lambda$ and $B \in \mathcal{V}_{SI}$. B is splitting in $\mathcal{V}$ iff there is a monolithic polynomial p for B. In this case $p = 1$ is a conjugate identity for B.

Proof:

Assume that B is splitting in $\mathcal{V}$. Then B has a conjugate identity $r = s$. Put

$t = r * s$. Then $t = 1$ is a conjugate identity for B. $B \models t = 1$ so there is a cardinal λ with $t \in W(\lambda)$ and an epimorphism $f\colon W(\lambda) \to B$ such that $f(t) \neq 1$. Then $f(t) < 1$ and $f(t) \in L(B)$. Hence $f(t) \leq m_B$. Now there is a $u \in W(\lambda)$ with $f(u) = m_B$. Put $p = u + t$. Then $t \vdash p$ and so the variety satisfying $p = 1$ contains the conjugate variety of B. Also $f(p) = m_B$ so $B \models p = 1$ whence the variety satisfying $p = 1$ is contained in the conjugate variety of B. Thus $p = 1$ is in fact a conjugate identity for B and $f(p) = m_B$. It remains to show that if $q \in W(\lambda)$ with $f(q) = m_B$ then $q \vdash p$. For this we note that if $f(q) = m_B$ then $B \models q = 1$. Hence the variety satisfying $q = 1$ is contained in the conjugate variety which is just the variety satisfying $p = 1$. Thus $q \vdash p$ as required.

Now suppose we have a monolithic polynomial $p \in W(\lambda)$ for B. Let $f\colon W(\lambda) \to B$ be the epimorphism as in Definition 2.18. Let $\mathscr{Z}$ be the variety satisfying $p = 1$. Since $f(p) = m_B$, $B \notin \mathscr{Z}$. Suppose there is a $\mathscr{W} \in (\mathscr{Z}] \cap [\mathrm{Var}\,\{B\})$ in $\Lambda_{\mathscr{V}}$. Then $B \in \mathscr{W} \subseteq \mathscr{Z}$, a contradiction and so $(\mathscr{Z}] \cap [\mathrm{Var}\,\{B\}) = \phi$. Let $\mathscr{W} \in \Lambda_{\mathscr{V}}$ and suppose $\mathscr{W} \notin [\mathrm{Var}\,\{B\})$. Then $B \notin \mathscr{W}$. Let $g\colon W(\lambda) \to \mathscr{F}(\mathscr{W},\lambda)$ be the canonical epimorphism. Then $\ker g \subseteq \ker f$ and so ther are $r,s \in \mathscr{W}(\lambda)$ with $(r,s) \in \ker_S$ but $(r,s) \notin \ker f$. Put $q = r * s$. Then $g(q) = 1$ but $f(q) < 1$. Now $f(q) \in L(B)$ so $f(q) \leq m_B$. Hence $f(q + p) = m_B$ and so $q + p \vdash p$. But $q \vdash q + p$ and so $q \vdash p$. Hence the variety satisfying $q = 1$ is contained in $\mathscr{Z}$. Since $g(q) = 1$, $\mathscr{W}$ is contained in the former so $\mathscr{W} \subseteq \mathscr{Z}$. Thus $\Lambda_{\mathscr{V}} = (\mathscr{Z}] \cup [\mathrm{Var}\,\{B\})$. It follows that B is splitting in $\mathscr{V}$ with conjugate identity $p = 1$. □

We now investigate methods of computing equational bases for interior algebra varieties.

Recall that if $\mathscr{V} \subseteq \mathscr{W}$ in Λ and $\mathscr{V}$ is finitely based relative to $\mathscr{W}$ then $\mathscr{V}$ is in fact one–based relative to $\mathscr{W}$. Given an equational base for $\mathscr{V}$ relative to $\mathscr{W}$ we can easily compute an (open) polynomial r such that $\{r = 1\}$ is a base for $\mathscr{V}$ relative to $\mathscr{W}$.

Proposition 2.20

If $\mathscr{V}, \mathscr{W}$ are as above and $\{p_1 = q_1, \ldots, p_n = q_n\}$ is a base for $\mathscr{V}$ relative to $\mathscr{W}$ put $r = \prod_{i=1}^{n} (p_i * q_i)$. Then $\{r = 1\}$ is a base for $\mathscr{V}$ relative to $\mathscr{W}$. □

Recall that if $\mathscr{V}, \mathscr{W} \subseteq \mathscr{Z}$ in Λ and $\mathscr{V}, \mathscr{W}$ are finitely based relative to $\mathscr{Z}$ then $\mathscr{V} + \mathscr{W}$ is finitely based relative to $\mathscr{Z}$ (Corollary 2.2). Of course $\mathscr{V} \cup \mathscr{W}$ is also finitely based relative to $\mathscr{Z}$. Given finite bases for $\mathscr{V}$ and $\mathscr{W}$ we can compute single identities defining $\mathscr{V} \cap \mathscr{W}$ and $\mathscr{V} + \mathscr{W}$ using Proposition 2.20 and the following :

Proposition 2.21

Let $\mathscr{V}, \mathscr{W} \subseteq \mathscr{Z}$ in Λ. Let p, q be open polynomials such that $\{p = 1\}$ and $\{q = 1\}$ are bases relative to $\mathscr{Z}$ for $\mathscr{V}$ and $\mathscr{W}$ respectively.

(i) $\{pq = 1\}$ is a base for $\mathscr{V} \cap \mathscr{W}$ relative to $\mathscr{Z}$.

(ii) If p, q have disjoint variable sets then $\{p = q = 1\}$ is a base for $\mathscr{V} + \mathscr{W}$ relative to $\mathscr{Z}$.

Proof:

(i) is clear. For (ii): Let $B \in (\mathscr{V} + \mathscr{W})_{FSI}$. Then by congruence distributivity $B \in \mathscr{V}_{FSI}$ or $B \in \mathscr{W}_{FSI}$. Hence $B \models p = 1$ or $B \models q = 1$ whence $B \models p + q = 1$. Now suppose B is F.S.I. and $B \models p = 1$ and $B \models q = 1$. Then since p, q have disjoint variable sets we can find a valuation making p, q simultaneously less than 1 in B. Now B is F.S.I. So 1 is join irreducible in $L(B)$. Hence under this valuation $p + q$ is less than 1 in B, contradicting $B \models p + q = 1$. Hence $B \models p = 1$ or $B \models q = 1$ whence $B \in (\mathscr{V} + \mathscr{W})_{FSI}$. It follows that $\{p + q = 1\}$ is a base for $\mathscr{V} + \mathscr{W}$ relative to $\mathscr{Z}$. □

Example 2.22

For $n < \omega$ let $\mathscr{V}_n$ denote the variety generated by S.I. interior algebras with at most n elements. An S.I. interior algebra has this property iff it satisfies the identity

$\sum_{i<j\leq n} (x_i * x_j) = 1$ and so this identity defines $\mathscr{V}_n$.

For $n < w$, $\mathscr{V}(I_n) = \mathscr{C} \cap \mathscr{V}_n$ and so $\{x^{IC} = x^I, \sum_{i<j\leq n} (x_i * x_j) = 1\}$ is a base for $\mathscr{V}(I_n)$, □

Example 2.23

$\sum_{i<j\leq n} (x_i^I * x_j^I) = 1$ is the identity defining the variety of interior algebras generated by S.I. interior algebras with at most n open elements.

Thus $\sum_{i<j\leq 4} (x_i * x_j) \cdot \sum_{k<\ell<3} (x_k^I * x_\ell^I) = 1$ is an identity defining $\mathscr{V}(S)$. □

3. HEYTING VARIETIES AND THE CARDINALITY OF Λ

We now show that there are $2^{\aleph_0}$ interior algebra subvarieties which are not generated by their finite (S.I.) members.

Definition 3.1

Let B be an interior algebra. We can define an implication (relative pseudocomplementation) on $L(B)$ by $a \rightarrow b = (a' + b')^I$. Then we obtain a Heyting algebra $\mathrm{Ha}\, B = \langle L(B), \cdot, +, \rightarrow, 0, 1\rangle$. If $f\colon B \rightarrow C$ is a homomorphism then $\mathrm{Ha}\, f = f|_{L(B)}$ is a Heyting algebra homomorphism. □

Let **Halg** denote the category of Heyting algebras and Heyting algebra homomorphisms.

Corollary 3.2

Ha : **Int** → **Halg** is a functor. □

The functor Ha has all the preservation properties of L :

Proposition 3.3

Ha preserves embeddings, epimorphisms, isomorphisms, products, subdirect products, direct limits and inverse limits. □

Like L , Ha also fails to be full and faithful. (See Example 1.1.6 in [7]). However Ha has some extra useful preservation properties :

Proposition 3.4

(i) Ha preserves congruence lattices i.e. for all B, $\mathrm{Con}(\mathrm{Ha}\, B) \cong \mathrm{Con}(B)$.

(ii) Ha preserves and reflects simple, S.I. and F.S.I. algebras.

Proof:

(i) Clear since Con (Ha B) is isomorphic to the lattice of filters in Ha B which is just FL(B) $\cong$ Con (B).

(ii) Clear from (i). □

Another important property of Ha is the following :

Proposition 3.5

Ha : **Int** $\rightarrow$ **Halg** is isomoprhism dense (i.e. every Heyting algebra is isomorphic to Ha (B) for some B). □

The above is just a restatement of the representation theorem for Heyting algebras : Every Heyting algebra is isomorphic to the Heyting algebra of open sets of a topological field of sets.

We are going to use the functor Ha to produce varieties of interior algebras.

Definition 3.6

Let $\mathscr{L}_I$ and $\mathscr{L}_H$ denote the first order languages for interior algebras and Heyting algebras respectively. We define an interpretation from $\mathscr{L}_H$ to $\mathscr{L}_I$ as follows :

For any variable x of $\mathscr{L}_H$ put $I(x) := x^I$, put $I(0) := 0$ and $I(1) := 1$. If t, s are terms such that $I(t)$, $I(s)$ have been defined put :

$I(t \,.\, s) := I(t) \,.\, I(s)$

$I(t + s) := I(t) + I(s)$

$I(t \rightarrow s) := (I(t)' + I(s))^I$.

This defines the interpretation on terms. Now if t, s are terms define :

$I(t = s) := I(t) = I(s)$

If φ, ψ are formulae such that $I(\varphi)$ and $I(\psi)$ have been defined, define :

$I(\varphi \wedge \psi) := I(\varphi) \wedge I(\psi)$

$I(\neg\varphi) := \neg I(\varphi)$

$I((\exists x)\varphi) := (\exists x)\, I(\varphi)$. □

Notice the similarity of the above with the interpretation of intuisionistic logic into modal logic.

Theorem 3.7

Let B be an interior algebra

(i) If $\varphi(x_1, ..., x_n)$ is a Heyting algebra formula and $a_1, ..., a_n \in L(B)$ then Ha B $\models \varphi[a_1, ..., a_n]$ iff B $\models I(\varphi)\, [a_1, ..., a_n]$.

(ii) If φ is a Heyting algebra sentence then Ha B $\models \varphi$ iff B $\models I(\varphi)$. □

The above theorem is easily proved by induction on the complexity of formulas.

Definition 3.8

For all isomorphism closed classes $\mathscr{K}$ of Heyting algebras let $I(\mathscr{K}) = Ha^{-1}[\mathscr{K}]$. If Σ is a set of $\mathscr{L}_H$ formulas let $I(\Sigma) = \{I(\varphi) : \varphi \in \Sigma\}$. □

Corollary 3.9

(i) The assignment $\mathscr{K} \longrightarrow I(\mathscr{K})$ is injective.

(ii) If Σ is a set of sentences of $\mathcal{L}_H$ and $\mathcal{K}$ denotes the variety of all Heyting algebras then $I(\mathcal{K} \cap \mathrm{Mod}\,\Sigma) = \mathcal{I} \cap \mathrm{Mod}\, I(\Sigma)$.

(iii) If $\mathcal{K}$ is an elementary class of Heyting algebras defined by sentences of a certain structural type then $I(\mathcal{K})$ is an elementary class of interior algebras defined by sentences of the same structural type.

(iv) If $\mathcal{K}$ is a strictly elementary class of Heyting algebras then $I(\mathcal{K})$ is a strictly elementary class of interior algebras.

Proof:

(i) follows from the isomorphism denseness of Ha. (ii) follows from Theorem 2.7 (ii). Notice that I changes the structure of terms but it does not change the way formulas are built up from atomic formulas. Thus (iii) follows from (ii) as does (iv). □

In particular if $\mathcal{V}$ is a variety of Heyting algebras $I(\mathcal{V})$ is a variety of interior algebras, a **Heyting variety**. Moreover $I(\mathcal{V})$ is uniquely determined by $\mathcal{V}$. For example $\mathcal{C}$ is a Heyting variety, it corresponds to the variety of Boolean algebras considered as a Heyting algebra variety.

Recall that there are $2^{\aleph_0}$ Heyting algebra varieties which are not generated by their finite (S.I.) members (see [2]).

Theorem 3.10

There are $2^{\aleph_0}$ Heyting varieties which are not generated by their finite (S.I.) members. In particular $|\Lambda| = 2^{\aleph_0}$.

Proof:

Suppose $\mathcal{V}$ is a variety of Heyting algebras which is not generated by its finite (S.I.) members. Now Ha preserves epimorphisms, embeddings and products, and Ha **B** is finite if **B** is finite. Hence $I(\mathcal{V})$ cannot be generated by its finite (S.I.) members. The result now follows from the remark preceding the theorem. □

Proposition 3.11

Let $\{\mathcal{V}_i : i \in I\}$ be a family of Heyting varieties. Then $\bigcap_{i \in I} \mathcal{V}_i$ is a Heyting variety.

Proof:

For all $i \in I$ there is a set Σ_i of Heyting algebra identities such that $\mathcal{I} \cap \mathrm{Mod}\, I(\Sigma_i) = \mathcal{V}_i$. Then $\bigcap_{i \in I} \mathcal{V}_i = \mathcal{I} \cap \bigcap_{i \in I} \mathrm{Mod}\, I(\Sigma_i) = \mathcal{I} \cap \mathrm{Mod} \bigcup_{i \in I} I(\Sigma_i) = \mathcal{I} \cap \mathrm{Mod}\, I(\bigcap_{i \in I} \Sigma_i)$ and so $\bigcap_{i \in I} \mathcal{V}_i$ is a Heyting variety. □

Proposition 3.12

The Heyting varieties form a sublattice of Λ.

Proof:

Let $\mathcal{V}, \mathcal{W} \in \Lambda$ be Heyting varieties. Then by Proposition 3.11 $\mathcal{V} \cap \mathcal{W}$ is a Heyting variety. Now there are Heyting algebra varieties $\mathcal{V}'$ and $\mathcal{W}'$ such that $\mathcal{V} = I(\mathcal{V}')$ and $\mathcal{W} = I(\mathcal{W}')$. Then by Proposition 3.4 (ii) $\mathcal{V}_{SI} = I(\mathcal{V}'_{SI})$ and $\mathcal{W}_{SI} = I(\mathcal{W}'_{SI})$. Then by congruence distributivity $(\mathcal{V} + \mathcal{W})_{SI} = \mathcal{V}_{SI} \cup \mathcal{W}_{SI} = I(\mathcal{V}'_{SI}) \cup I(\mathcal{W}'_{SI}) = I(\mathcal{V}'_{SI} \cup \mathcal{W}'_{SI}) = I(\mathcal{V}' + \mathcal{W}')_{SI})$. Thus $\mathcal{V} + \mathcal{W} = I(\mathcal{V}' + \mathcal{W}')$ and so $\mathcal{V} + \mathcal{W}$ is a Heyting variety. □

Note that although the Heyting varieties form a complete lattice, they do not form a complete sublattice of Λ.

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Department of Mathematics
University of Cape Town
Rondebosch 7700
South Africa